ASCHAM SCHOOL

MATHEMATICS EXTENSION 2

TRIAL EXAMINATION

2002

Time : 3 hours + 5 minutes reading time

Instructions:

Attempt all questions

All questions are of equal value

All necessary working should be shown for every question.

Full marks may not be awarded for careless or badly arranged work

A Table of Standard Integrals is provided

Approved calculators may be used

Each question should be answered in a separate booklet

Question 1

(a) Let $z = -1 - \sqrt{3}\, i$.

(i) Write z in modulus-argument form. [2]

(ii) Show that z^6 is a real number. [2]

(b) (i) Simplify $\left(\sqrt{3} + \sqrt{3}i\right)^2$ [1]

(ii) Solve $z^2 - (1-i)z - 2i = 0$ writing the solutions in the form x + iy, where x and y are real. [3]

(c) Sketch the region in the complex number plane in which the following inequalities all hold:

$|z - 4| < |z - 4i|$ and $|z - 4| \le 4$ and $0 \le \arg(z - 4) < \dfrac{3\pi}{4}$ [4]

(d) Vertex A of square ABCD is represented by the complex number 5 + 2i and its centre X is represented by 2 + i. Find, in the form a + ib where a and b are real, the complex numbers representing the other three vertices. [3]

Question 2

(a) Find (i) $\int \sec^2 x \tan x \, dx$ by letting $u = \sec x$ [2]

(ii) $\displaystyle\int \frac{dx}{\sqrt{x^2 - 6x + 5}}$ by completing the square [2]

(iii) $\displaystyle\int \frac{dx}{5 + 3\cos x}$ by substituting $t = \tan\dfrac{x}{2}$ [4]

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sin 5x \cos 3x dx$ [3]

(c) (i) If $I_n = \int \tan^n x dx$ for integral $n \ge 2$ show that $I_n = \dfrac{1}{n-1}\tan^{n-1}\theta - I_{n-2}$. [2]

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^4 x dx$ [2]

Question 3

a) Show that if α,β and γ are the roots of the equation $x^3+px^2+qx+r=0$ and that $\alpha\beta+1=0$ then $1+q+pr+r^2=0$. [3]

b) (i) Prove that 1 and −1 are zeroes of multiplicity 2 of the polynomial x^6-3x^2+2. Hence express x^6-3x^2+2 as a product of irreducible factors over the field of : [2]

(α) rational numbers [1]

(β) complex numbers [2]

c) (i) Express cos5θ as a polynomial in terms of cosθ. [3]

(ii) Hence show that $x=\cos\frac{2k}{5}\pi$ for k = 0, 1, 2, 3, 4 are roots of the equation $16x^5-20x^3+5x-1=0$ and prove that $\cos\frac{\pi}{5}\cos\frac{2\pi}{5}=\frac{1}{4}$. [4]

Question 4

a) Evaluate $\int_{-2}^{10} x\sqrt{6+x}\,dx$ [3]

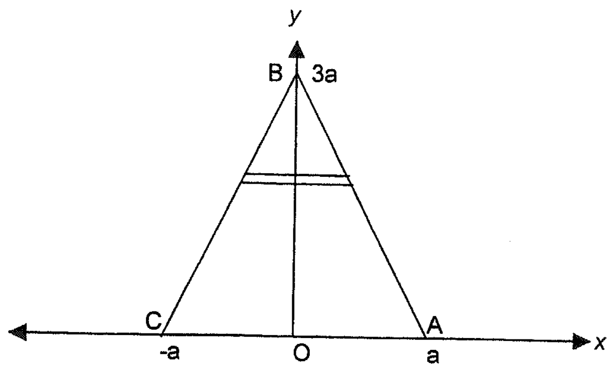
b) The foci of an ellipse are S (4,0) and S' (-4,0) and P is any point on the ellipse such that SP + S'P = 10. Find the equation of the ellipse. [4]

c) The hyperbola xy = 4 is rotated 45° clockwise about its centre. Find the equation of this hyperbola and sketch it labelling the vertices, foci, directrices and asymptotes. [5]

d) Solve cos 4x = sin 3x. [3]

Question 5

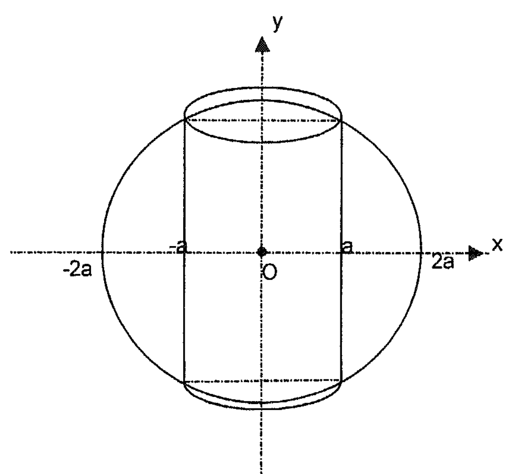
a)



(i) Find the equation of AB. [2]

(ii) Every cross-section perpendicular to OB is the base of a square. Find the volume of the solid formed with ABC as base. [5]

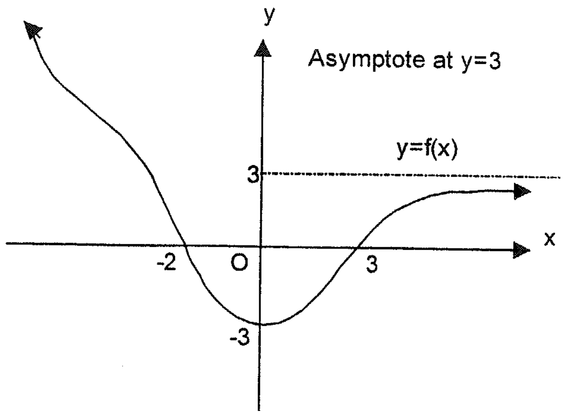
b) (i) Find the area of an ellipse with semi-major axis of length a units and semi-minor axis of length $\frac{1}{2}$a units. [2]



(ii) An elliptical hole with cross-section determined in (i) is bored symmetrically through a sphere of radius 2a units. Show the total volume remaining is $5\pi a^3\sqrt{3}$ cubic units. [6]

Question 6

a) The diagram shows the graph of y = f(x).



Sketch graphs of

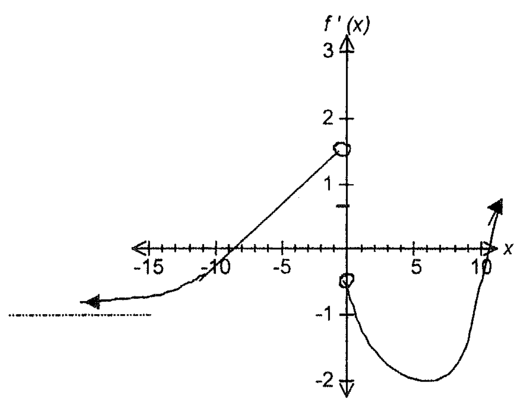
(i) $y = |f(x)|$ [1]

(ii) $y = \dfrac{1}{f(x)}$ [2]

(iii) $y^2 = f(x)$ [2]

(iv) the inverse function $y = f^{-1}(x)$ [2]

b)



The diagram is a sketch of y = f '(x) with a horizontal asymptote at y = -1.
Sketch y = f(x) given that it is continuous and f(-15) = f(5) =0, clearly labelling important features. [3]

c) ABCD is a cyclic quadrilateral and the opposite sides AB and DC are produced to meet at P, and the sides CB and DA meet at Q. If the two circles through the vertices of the triangles PBC and QAB intersect at R:

(i) Draw a diagram showing this information. [1]

(ii) Prove that P, R and Q are collinear. [3]

(iii) Explain why triangle PBQ can never be isosceles. [1]

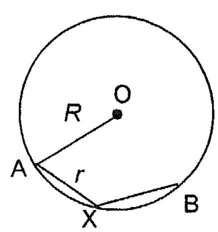
Question 7

a) Solve $\tan^{-1}x + \tan^{-1}(1-x) = \tan^{-1}\frac{9}{7}$, for x . [3]

b) (i) Prove that if x,y and z are positive $x^2+y^2+z^2 \geq xy+yz+xz$ [2]

(ii) If x,y and z are positive with constant sum k, show that the least value of $x^2+y^2+z^2$ is $\frac{1}{3}k^2$ [3]

c)



A and B are points on the circumference of a circular pond, centre O of radius R. A toy yacht is tied by means of a string of length r ($r<2R$) to a point X on the circumference of the pond such that the points A and B are the farthest points of the circumference of the pond that the yacht can reach. If $\angle AOX = \theta$ radians

Prove that: (i) $\angle AXB = (\pi - \theta)$ [2]

(ii) $r = 2R\sin\frac{1}{2}\theta$ [1]

(iii) the area of the pond in which the yacht can sail is $R^2[\pi - (\pi-\theta)\cos\theta - \sin\theta]$ [4]

Question 8

a) Use mathematical induction to show that

$$\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots\ldots\ldots\ldots\ldots+\sqrt{n} \le \frac{4n+3}{6}\sqrt{n} \text{ for all integers } n \ge 1. \qquad [5]$$

b) A particle moving in a straight line from the origin is subject to a resisting force which produces a retardation of kv^3 where v is the speed at time t and k is a constant. If u is the initial speed, x is the distance moved in time t,

(i) Show that $v = \dfrac{u}{kux+1}$. [3]

(ii) Deduce that $kx^2 = 2t - \dfrac{2x}{u}$. [2]

(iii) A bullet is fired horizontally at a target 3,000m away. The bullet is observed to take 1 second to travel the first 1,000m and 1.25seconds to travel the next 1,000m. Assuming that the air resistance is proportional to v^3, and neglecting gravity calculate the time taken to travel the last 1,000m. [5]

End of Examination

$= -1 - \sqrt{3}i$

$|z| = \sqrt{1+3}$ $\qquad \arg z = -\pi + \tan^{-1}\sqrt{3}$

$= 2$ $\qquad = -\pi + \frac{\pi}{3}$

$= -\frac{2\pi}{3}$

$2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

$= 2^6\left(\cos\left(-\frac{12\pi}{3}\right) + i\sin\left(-\frac{12\pi}{3}\right)\right)$ (by de Moivre)

$= 64\left[\cos(-4\pi) + i\sin(4\pi)\right]$

$= 64\left[1 + 0\right]$

$= 64$ which is a real no as required.

$(1 + \sqrt{3}i)^2 = 3 + 6i - 3$

$= 6i$

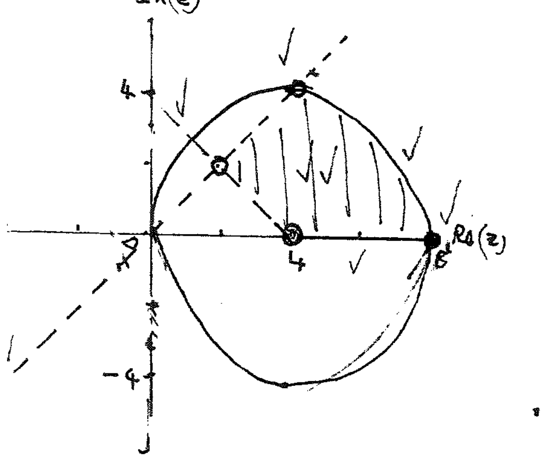
$z^2 - (1-i)z - 2i = 0$

$z = \frac{(1-i) \pm \sqrt{1 - 2i - 1 + 8i}}{2}$

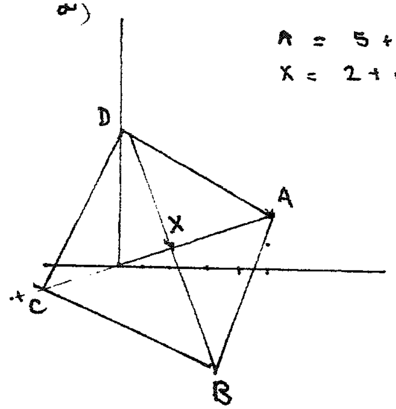
$= \frac{(1-i) \pm \sqrt{6i}}{2}$

$= \frac{(1-i) \pm (\sqrt{3} + \sqrt{3}i)}{2}$

$= \frac{(1+\sqrt{3}) + i(\sqrt{3}-1)}{2}$ or $\frac{(1-\sqrt{3}) - i(1+\sqrt{3})}{2}$



a)



$A = 5 + 2i$
$X = 2 + i$

$\vec{XA} = (5 + 2i) - (2 + i)$
$= 3 + i$

$\vec{XC} = -3 - i$

$\therefore C = -3 - i + 2 + i$
$= -1$

$\vec{XD} = i(3 + i)$
$= -1 + 3i$

$\therefore D = -1 + 3i + 2 + i$
$= 1 + 4i$

$\vec{XB} = 1 - 3i$

$\therefore B = 1 - 3i + 2 + i$
$= 3 - 2i$

Question 2

a) (i) $\int \sec^2 x \tan x \, dx$ $\qquad u = \sec x$, $du = \sec x \tan x \, dx$

$= \int u \, du$

$= \frac{1}{2}u^2 + c$

$= \frac{1}{2}\sec^2 x + c$

(ii) $\int \frac{dx}{\sqrt{x^2 - 6x + 5}}$

$= \int \frac{dx}{\sqrt{(x^2 - 6x + 9) - 4}}$

$= \int \frac{dx}{\sqrt{(x-3)^2 - 4}}$

$= \ln\left[x - 3 + \sqrt{x^2 - 6x + 5}\right] + C$

(iii) $\int \frac{dx}{5 + 3\cos x}$ $\qquad t = \tan\frac{x}{2}$, $dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$, $dx = \frac{2\,dt}{1+t^2}$, $\cos x = \frac{1 - t^2}{1 + t^2}$

$= \int \frac{\frac{2\,dt}{1+t^2}}{5 + 3\left(\frac{1-t^2}{1+t^2}\right)}$

$= \int \frac{2\,dt}{5 + 5t^2 + 3 - 3t^2}$

$= \int \frac{dt}{4 + t^2}$ $\qquad \therefore \frac{1}{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{2}\right) + c$

$= \frac{1}{4}\int \frac{1}{2-t} + \frac{1}{2+t}\,dt$ $\qquad \frac{1}{4 - t^2} = \frac{A}{2-t} + \frac{B}{2+t}$

$1 = A(2+t) + B(2$

$2A + 2B = 1$

$A - B = 0$

$= \frac{1}{4}\ln\left(\frac{2+t}{2-t}\right) + c$ $\qquad A = B = \frac{1}{4}$

b) $\int_0^{\frac{\pi}{4}} \sin 5x \cos 3x \, dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sin 8x + \sin 2x)\,dx$

$= \frac{1}{2}\left[-\frac{1}{8}\cos 8x - \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}}$

$= -\frac{1}{2}\left[\frac{1}{8}\cos 2\pi + \frac{1}{2}\cos\frac{\pi}{2}\right] + \frac{1}{2}\left[\frac{1}{8}\cos 0 + \frac{1}{2}\cos 0\right]$

$= -\frac{1}{2}\left[\frac{1}{8} - 0\right] + \frac{1}{2}\left[\frac{1}{8} + \frac{1}{2}\right]$

$= \frac{1}{4}$

c) (i) $I_n = \int \tan^n x \, dx$

$= \int \tan^{n-2} x (\sec^2 x - 1)\,dx$

$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$

$= \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$ as required

(ii) $\int_0^{\frac{\pi}{4}} \tan^4 x\,dx = \frac{1}{3}\left[\tan^3 x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^2 x\,dx$

$= \frac{1}{3}\left[1 - 0\right] - \int_0^{\frac{\pi}{4}} \sec^2 x - 1\,dx$

$= \frac{1}{3} - \left[\tan x - x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{3} - \left[1 - \frac{\pi}{4} - 0\right]$

$= \frac{3\pi - 8}{12}$

$x^3 + px^2 + qx + r = 0$

$\alpha + \beta + \gamma = -p$ (1)

$\alpha\beta + \alpha\gamma + \beta\gamma = q$ (2)

$\alpha\beta\gamma = -r$ (3)

Since $\alpha\beta + 1 = 0$ i.e. $\alpha\beta = -1$

$\alpha\gamma + \beta\gamma = q + 1$

$\gamma = r$

$r(\alpha + \beta) = q + 1$

$\alpha + \beta = -p - r$

$r(-p - r) = q + 1$

$1 + q + pr + r^2 = 0$

or Subst in $x^3 + px^2 + qx + r = 0$

$r^3 + pr^2 + qr + r = 0$

$r(1 + q + pr + r^2) = 0$

$r \neq 0$ or $1 + q + pr + r^2 = 0$

Let $P(x) = x^6 - 3x^2 + 2$

$P'(x) = 6x^5 - 6x$

$P(1) = P'(1) = 0$

$P(-1) = P'(-1) = 0$

1 and -1 are zeroes of multiplicity 2.

$x^6 - 3x^2 + 2$

$= (x-1)^2(x+1)^2(x^2 + ax + b)$

$= (x^2 - 1)^2(x^2 + ax + b)$

$= (x^4 - 2x^2 + 1)(x^2 + 2)$

$= (x - 1)^2(x+1)^2(x^2 + 2)$

$= (x - 1)^2(x+1)^2(x + i\sqrt{2})(x - i\sqrt{2})$

$\cos 5\theta = \cos(\theta + 4\theta)$

$= \cos\theta\cos 4\theta - \sin\theta\sin 4\theta$

$= \cos\theta(2\cos^2 2\theta - 1) - \sin\theta\,(2\sin 2\theta\cos 2\theta)$

$= \cos\theta\left[2(2\cos^2\theta - 1)^2 - 1\right] - 4\sin^2\theta\cos\theta(2\cos^2\theta - 1)$

$= \cos\theta\left[4\cos^4\theta - 4\cos^2\theta + 1\right] - \cos\theta - 8\sin^2\theta\cos^3\theta + 4\sin^2\theta\cos\theta$

$= 8\cos^5\theta - 8\cos^3\theta + 2\cos\theta - \cos\theta - 8\cos^3\theta + 8\cos^5\theta + 4\cos\theta - 4\cos^3\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(ii) $\cos 5\theta = 1$

$5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi, \ldots$

$\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

Replacing $\cos\theta$ with x in (i)

$16x^5 - 20x^3 + 5x = 1$

$16x^5 - 20x^3 + 5x - 1 = 0$

i.e. $\cos\left(\frac{2k}{5}\pi\right)$ where $k = 0, 1, 2, 3, 4$ are the roots of

$16x^5 - 20x^3 + 5x - 1 = 0$

Product of roots $= \frac{1}{16}$

$\cos 0 \cdot \cos\frac{2\pi}{5} \cdot \cos\frac{4\pi}{5} \cdot \cos\frac{6\pi}{5} \cdot \cos\frac{8\pi}{5} = \frac{1}{16}$

$1 \cdot \cos\frac{2\pi}{5} \cdot -\cos\frac{\pi}{5} \cdot -\cos\frac{\pi}{5} \cdot \cos\frac{2\pi}{5} = \frac{1}{16}$

$\cos^2\frac{2\pi}{5}\cos^2\frac{\pi}{5} = \frac{1}{16}$

$\cos\frac{2\pi}{5}\cos\frac{\pi}{5} = \frac{1}{4}$ $\quad \cos\frac{2\pi}{5}, \cos\frac{\pi}{5} > 0$

Question 4

a) $\int_{-2}^{10} x\sqrt{6 + x}\,dx$ $\quad$ Let $u = 6 + x$, $du = dx$

When $x = -2$, $u = 4$

When $x = 10$, $u = 16$

$= \int_4^{16} (u - 6)\sqrt{u}\,du$

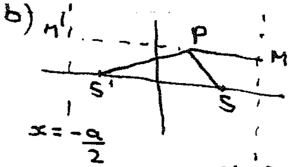
$= \int_4^{16} u^{3/2} - 6u^{1/2}\,du$

$= \left[\frac{2}{5}u^{5/2} - 4u^{3/2}\right]_4^{16}$

$= \frac{2}{5} \times 4^5 - 4 \times 64 - \frac{2}{5} \times 32 + 4 \times 8$

$= \frac{864}{5}$

b)



By definition

$PS = e\,PM$

$PS' = e\,PM'$

$PS + PS' = e(PM + PM')$

$= e\,MM'$

$= e \times 2\frac{a}{e}$

$= 2a$

$2a = 10$

$a = 5$

$S = (4, 0) = (ae, 0)$

$ae = 4$

$5e = 4$

$e = \frac{4}{5}$

$b^2 = a^2(1 - e^2)$

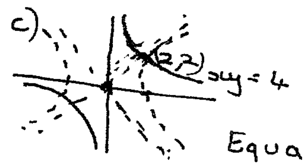
$= 25\left(1 - \frac{16}{25}\right)$

$= 9$

$b = 3$

$\therefore$ Ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$

c)

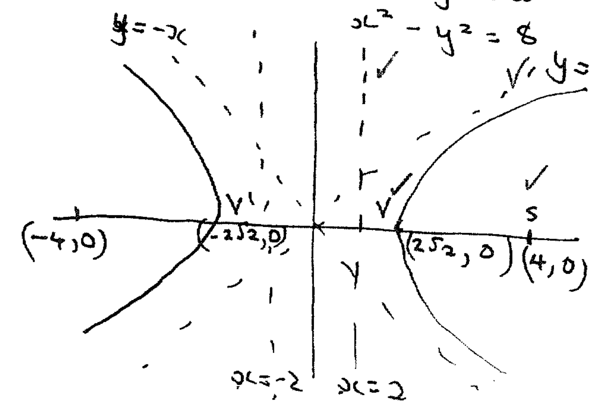


$a = \sqrt{2^2 + 2^2}$

$= 2\sqrt{2}$

Equation of hyperbola is $x^2 - y^2 = a^2$

$x^2 - y^2 = 8$



) $\cos 4x = \sin 3x$

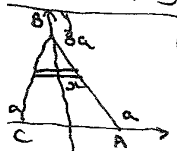
$= \cos\left(\frac{\pi}{2} - 3x\right)$

$4x = 2n\pi + \left(\frac{\pi}{2} - 3x\right)$ or $4x = 2n\pi - \left(\frac{\pi}{2} - 3x\right)$

$7x = 2n\pi + \frac{\pi}{2}$ or $x = 2n\pi - \frac{\pi}{2}$

$x = \frac{2n\pi}{2} + \frac{\pi}{14}$ where n is an integer

Question 5



(i) $m_{AB} = -\frac{3a}{a}$

$= -3$

$y = -3x + 3a$

$A(y) = (2x)^2$ where $x = \frac{3a-y}{3}$

$= \left[\frac{2}{3}(3a-y)\right]^2$

$= \frac{4}{9}(9a^2 - 6ay + y^2)$

$\Delta V = \frac{4}{9}(9a^2 - 6ay + y^2)\Delta y$

$V \doteqdot \sum_{y=0}^{3a} \frac{4}{9}(9a^2 - 6ay + y^2)\Delta y$

$= \lim_{\Delta y \to 0} \sum_{y=0}^{3a} \frac{4}{9}(9a^2 - 6ay + y^2)\Delta y$

$= \frac{4}{9}\int_0^{3a} 9a^2 - 6ay + y^2 \, dy$

$= \frac{4}{9}\left[9a^2y - 3ay^2 + \frac{1}{3}y^3\right]_0^{3a}$

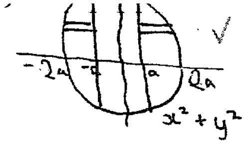
$= \frac{4}{9}\left[(27a^3 - 27a^3 + 9a^3) - 0\right]$

$= 4a^3$ cubic units

b)(i) Area of ellipse $= \pi ab$ where $a = a$, $b = \frac{a}{2}$

$= \pi \times a \times \frac{a}{2}$

$= \frac{\pi a^2}{2}$



$x^2 + y^2 = 4a^2$

$A(y) = \pi x^2 - \frac{\pi a^2}{2}$ where $x^2 = 4a^2 - y^2$

$= \pi\left(4a^2 - y^2 - \frac{a^2}{2}\right)$

$= \pi\left(\frac{7}{2}a^2 - y^2\right)$

When $x = a$: $a^2 + y^2 = 4a^2$

$y^2 = 3a^2$

$y = \pm a\sqrt{3}$

$\Delta V = \pi\left(\frac{7}{2}a^2 - y^2\right)\Delta y$

$V \doteqdot \sum_{y=-a\sqrt{3}}^{a\sqrt{3}} \pi\left(\frac{7}{2}a^2 - y^2\right)\Delta y$

$= \lim_{\Delta y \to 0} 2\pi \sum_{y=0}^{a\sqrt{3}} \left(\frac{7}{2}a^2 - y^2\right)\Delta y$

$= 2\pi\int_0^{a\sqrt{3}} \frac{7}{2}a^2 - y^2 \, dy$

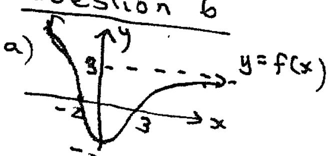
$= 2\pi\left[\frac{7}{2}a^2y - \frac{1}{3}y^3\right]_0^{a\sqrt{3}}$

$= 2\pi\left[\frac{7}{2}a^3\sqrt{3} - \frac{1}{3} \times 3a^3\sqrt{3}\right]$

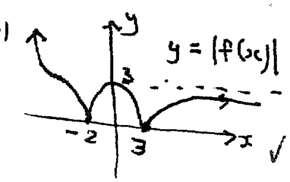
$= 5\pi\sqrt{3}\,a^3$ cubic units

Question 6

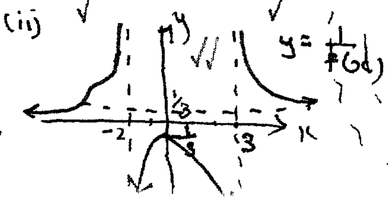
a)



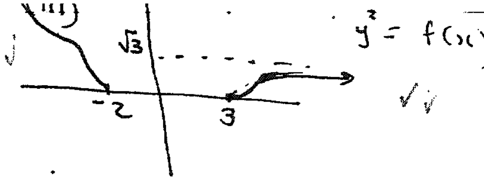
(i)



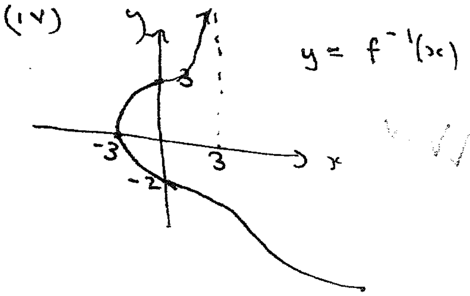
(ii)



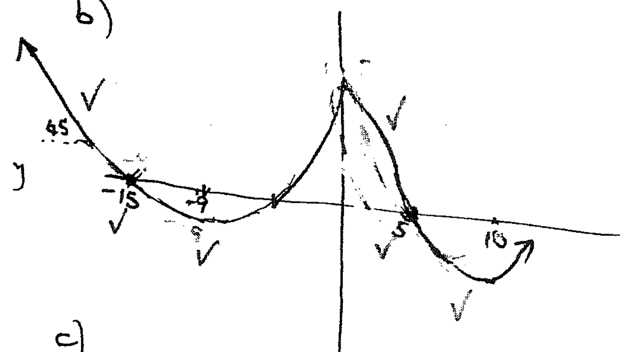
(iii)



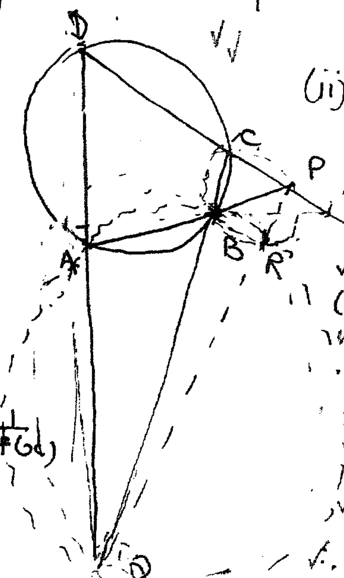
(iv)



b)



c) (i)



(ii)

let $\angle PRB = x$

$\angle DCB = x°$ (ext $\angle$ of cyclic quad PCBR)

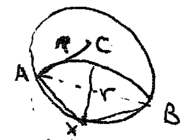
$\angle DAB = 180° - x$ (opp $\angle$'s of cyclic quad ABCD)

$\angle BRQ = 180 -$ (ext $\angle$ of cyclic quad ABRQ)

$\angle PRB + \angle BRQ = 180°$

$\therefore$ P, R, Q are collinear

(iii)



Area yacht can sail

$\therefore$ area of segment ACB + area segment AXB

$= \frac{1}{2} r^2 [(\pi - \theta) - \sin(\pi-\theta)] + \frac{1}{2} R^2 [2\theta - \sin 2\theta]$

$= \frac{1}{2} \times 4R^2 \sin^2 \frac{1}{2}\theta \; [\pi - \theta - \sin\theta] + \frac{1}{2}R^2 [2\theta - \sin 2\theta]$

$= 2R^2 \sin^2 \frac{1}{2}\theta [\pi - \theta - \sin\theta] + \frac{1}{2}R^2 [2\theta - 2\sin\theta\cos\theta]$

$= R^2 (1 - \cos\theta)(\pi - \theta - \sin\theta] + R^2 [\theta - \sin\theta\cos\theta)$

$= R^2 [\pi - \theta - \sin\theta - (\pi - \theta)\cos\theta + \sin\theta\cos\theta + \theta - \sin\theta\cos\theta]$

$= R^2 [\pi - \sin\theta - (\pi - \theta)\cos\theta]$ as required

Question 7

$\tan^{-1} x + \tan^{-1} (1-x) = \tan^{-1} \frac{9}{7}$

$\tan^{-1} x = \alpha$ and $\tan^{-1} (1-x) = \beta$

$\alpha + \beta = \tan^{-1} \frac{9}{7}$

$\tan(\alpha+\beta) = \frac{9}{7}$

$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{9}{7}$

$\frac{x + (1-x)}{1 - x(1-x)} = \frac{9}{7}$

$\frac{1}{1 - x + x^2} = \frac{9}{7}$

$9x^2 - 9x + 9 = 7$

$9x^2 - 9x + 2 = 0$

$(3x - 1)(3x - 2) = 0$

$x = \frac{1}{3}$ or $x = \frac{2}{3}$

)

) $\angle AOX = \theta$

$\angle XOB = \theta \quad (AX = XB)$

obtuse $AOB = 2\pi - 2\theta$ (revolution)

$\angle AXB = \pi - \theta$ ($\angle$ at circumference $= \frac{1}{2}$ $\angle$ at centre on major arc AB)

) In Δ OAX $\angle OAX = \frac{\pi - \theta}{2}$ ($\angle$ sum of isosc Δ; OA = OB)

by Sine Rule

$\frac{r}{\sin\theta} = \frac{R}{\sin(\frac{\pi}{2} - \frac{\theta}{2})}$

$r = \frac{R\sin\theta}{\cos\frac{\theta}{2}}$

$= \frac{2R\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}}$

c) (i) For all real x, y, z

$(x - y)^2 \geq 0$

$x^2 + y^2 \geq 2xy$

Similarly $y^2 + z^2 \geq 2zy$

$x^2 + z^2 \geq 2xz$

Adding $2(x^2 + y^2 + z^2) \geq 2(xy + zy + xz)$

$\therefore x^2 + y^2 + z^2 \geq xy + zy + xz$

(ii) $x + y + z = k$

$(x + y + z)^2 = k^2$

$x^2 + y^2 + z^2 + 2(xy + zy + zx) = k^2$

$x^2 + y^2 + z^2 + 2(x^2 + y^2 + z^2) \geq k^2$

$3(x^2 + y^2 + z^2) \geq k^2$

$\therefore x^2 + y^2 + z^2 \geq \frac{1}{3}k^2$

$\therefore$ Least value of $x^2 + y^2 + z^2$ is $\frac{1}{3}k^2$

) To prove $\sqrt{1} + \sqrt{2} + \sqrt{3} + \dots + \sqrt{n} \leq \frac{4n+3}{6}\sqrt{n}$
for integers $n \geq 1$

When $n = 1$

$LHS = \sqrt{1}$ $\quad$ $RHS = \frac{4+3}{6}\sqrt{1}$

$= 1$ $\quad$ $= \frac{7}{6}$

$LHS < RHS$

$\therefore$ Statement is true for $n = 1$ ✓✓

Assuming true for $n = k$

$\sqrt{1} + \sqrt{2} + \dots + \sqrt{k} \leq \frac{4k+3}{6}\sqrt{k}$

... true for $n = k+1$

...rove

$... + \sqrt{2} + \dots + \sqrt{k} + \sqrt{k+1} \leq \frac{4k+7}{6}\sqrt{k+1}$ ✓✓

$... = \sqrt{1} + \sqrt{2} + \dots + \sqrt{k} + \sqrt{k+1}$

$\leq \frac{4k+3}{6}\sqrt{k} + \sqrt{k+1}$ ✓

...essary to prove

$\frac{...k+7}{6}\sqrt{k+1} \leq \frac{4k+3}{6}\sqrt{k} + \sqrt{k+1}$

...der the difference ✓

$\frac{...+7}{6}\sqrt{k+1} - \frac{4k+3}{6}\sqrt{k} - \sqrt{k+1}$

$\frac{...+k+1}{6}\sqrt{k+1} - \frac{4k+3}{6}\sqrt{k}$

$\frac{4k+1}{6}\sqrt{k+1} \div \frac{4k+3}{6}\sqrt{k+1}$ $\quad$ since $k \geq$ ✓✓ so $\sqrt{k+1} \geq \sqrt{k}$

$-\frac{1}{3}\sqrt{k+1}$

0 $\quad$ since $\sqrt{k+1} > 0$

$\frac{4k+7}{6}\sqrt{k+1} < \frac{4k+3}{6}\sqrt{k} + \sqrt{k+1}$

$\sqrt{1} + \sqrt{2} + \dots + \sqrt{k} + \sqrt{k+1} \leq \frac{4k+7}{6}\sqrt{k+1} \leq \frac{4k+3}{6}\sqrt{k} + \sqrt{k+1}$

Statement is true for $n = k+1$ when it is true for $n = k$

By steps (A) and (B) by math. induction statement is true for $n \geq 1$ ✓✓

b) $\ddot{x} \longrightarrow$ $\quad$ $\longleftarrow mkv^3$

(i) $m\ddot{x} = -mkv^3$

$\ddot{x} = -kv^3$ ✓

$v\frac{dv}{dx} = -kv^3$ ✓

$\frac{dv}{dx} = -kv^2$

$\frac{dx}{dv} = -\frac{1}{k}v^{-2}$ ✓

$x = \frac{1}{k}v^{-1} + c$

When $x = 0$ $v = u$

$0 = \frac{1}{ku} + c$

$c = -\frac{1}{ku}$ ✓

$x = \frac{1}{kv} - \frac{1}{ku}$

$\frac{1}{kv} = x + \frac{1}{ku}$ ✓

$= \frac{kux + 1}{ku}$

$kv = \frac{ku}{kux + 1}$

$v = \frac{u}{kux + 1}$ ✓

(ii) $\frac{dx}{dt} = \frac{u}{kux+1}$

$\frac{dt}{dx} = kx + \frac{1}{u}$ ✓

$t = \frac{1}{2}kx^2 + \frac{x}{u} + c$

When $t = 0$ $x = 0$ $c = 0$ ✓

$kx^2 = 2t - \frac{2x}{u}$ ✓

(iii) $kx^2 = 2t - \frac{2x}{u}$

When $t = 1$ $\quad$ $x = 1000$

$k \times 10^6 = 2 - \frac{2 \times 10^3}{u}$ ✓✓

$k = 2 \times 10^{-6} - \frac{2}{u} \times 10^{-3}$

When $t = 2.25$ $\quad$ $x = 2000$

$k \times 4 \times 10^6 = 4.5 - \frac{4 \times 10^3}{u}$

$k = 1.125 \times 10^{-6} - \frac{10^{-3}}{u}$ ✓✓

$\therefore$ $2 \times 10^{-6} - \frac{2}{u} \times 10^{-3} = 1.125 \times 10^{-6} - \frac{10^{-3}}{u}$

$10^{-3}\left(\frac{2}{u} - \frac{1}{u}\right) = 0.875 \times 10^{-6}$

$\frac{10^{-3}}{u} = 8.75 \times 10^{-7}$

$u = \frac{10^{-3}}{8.75 \times 10^{-7}}$

$= 1142\frac{6}{7}$ ✓✓

1142.7

$= 2.5 \times 10^{-7}$ ✓✓

then $\cancel{x} = 3000$

$.5 \times 10^{-7} \times 9 \times 10^{6} = 2t - \frac{6000}{1142\frac{6}{7}}$

$2t = 7\frac{1}{2}$

$t = 3.75\ s$ ✓

Time taken for last 1000 m is $3.75 - 2.25\ s$.

ie. $1.5\ s$ ✓